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Precise Time-Step Integration Algorithms Using Response Matrices with Expanded Dimension

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In this paper, the precise time-step integration method by step-response and impulsive-response matrices is further developed by expanding the dimension of the matrices so as to avoid computing the particular solutions separately. Two new precise time-step integration algorithms with excitations described by second-order and first-order differential equations are proposed. The first method is a direct extension of the existing algorithms. However, the extended system matrices are not symmetrical. In the second method, the Duhamel integrals are used as the particular solutions. As a result, the responses can be expressed in terms of the given initial conditions and the step-response matrix, the impulsive-response matrix and a newly derived Duhamel-response matrix. The symmetry property of the system matrices can be used in the computation. However, it will first require a calculation of the Duhamel-response matrix and its derivative. To reduce the computational effort, the relation between the Duhamel-response matrix and its derivative is established. A special computational procedure for periodic excitation is also discussed. Numerical examples are given to illustrate the present highly efficient algorithms.

Nomenclature

$[\mathbf{C}]$	=	time-invariant damping matrix
$[\mathbf{D}(t)]$	=	Duhamel-response matrix
$[\dot{\mathbf{D}}(t)]$	=	time derivative of Duhamel-response matrix
$\mathrm{d}/\mathrm{d}t$	=	differentiation with respect to time t
$[\mathbf{E}(t)]$	=	$\exp([\mathbf{S}_1]t)$
$[\mathbf{f}]$	=	coefficient matrix describing the external excitation
$[\mathbf{G}(t)]$	=	step-response matrix
$[\mathbf{G}^*(t)]$	=	extended step-response matrix with expanded dimension
$[\dot{\mathbf{G}}(t)]$	=	time derivative of step-response matrix
$[\dot{\mathbf{G}}^*(t)]$	=	time derivative of extended step-response matrix
g	=	number of terms used to describe the excitations
$[\mathbf{H}(t)]$	=	impulsive-response matrix
$[\mathbf{H}^*(t)]$	=	extended impulsive-response matrix with expanded dimension
$[\dot{\mathbf{H}}(t)]$	=	time derivative of impulsive-response matrix
$[\dot{\mathbf{H}}^*(t)]$	=	time derivative of extended impulsive-response matrix
$[\mathbf{I}]$	=	identity matrix
$(Inte)$	=	computational cost for evaluating particular solutions
$[\mathbf{J}(t)]$	=	time derivative of impulsive-response matrix, $=[\dot{\mathbf{H}}(t)]$
$[\mathbf{J}^*(t)]$	=	time derivative of extended impulsive-response matrix, $=[\dot{\mathbf{H}}^*(t)]$
$[\mathbf{J}_a(t)]$	=	$[\mathbf{J}(t)]-[\mathbf{I}]$
$[\mathbf{J}_a^*(t)]$	=	$[\mathbf{J}^*(t)]-[\mathbf{I}]$
$[\mathbf{K}]$	=	time-invariant stiffness matrix
$[\mathbf{M}]$	=	time-invariant mass matrix
m	=	number of recursive evaluations in the scaling and squaring method
N_s	=	dimension of system
n	=	number of terms in the Taylor series approximation
$\{\mathbf{r}(t)\}$	=	external excitation vector
$\{\bar{\mathbf{r}}(t)\}$	=	external excitation vector with time shift
$[\mathbf{S}_1]$	=	coefficient matrix for excitation described by first-order differential equations
$[\mathbf{S}_2]$	=	coefficient matrix for excitation described by second-order differential equations
T	=	time period
$\{\mathbf{u}(t)\}$	=	time-dependent displacement vector
$\{\mathbf{u}_0\}$	=	initial displacement vector
$\{\mathbf{u}_s(t)\}$	=	steady-state displacement response
$\{\mathbf{v}(t)\}$	=	time-dependent velocity vector
$\{\mathbf{v}_0\}$	=	initial velocity vector
$\{\mathbf{v}_s(t)\}$	=	steady-state velocity response
$\{\mathbf{Z}(t)\}$	=	containing time functions describing the external excitation
Δt	=	time-step size
$[\mathbf{0}]$	=	zero matrix

I. Introduction

DYNAMIC responses of structures subjected to transient loading can be obtained by direct integration schemes. Many time-step integration methods [1] are available. The time-step size of these methods must be chosen carefully relative to the natural periods of the structures and the time variation of the excitations to evaluate the responses accurately.

The governing equation of a discretized structural model can be written as

$$[\mathbf{M}]\{\ddot{\mathbf{u}}(t)\}+[\mathbf{C}]\{\dot{\mathbf{u}}(t)\}+[\mathbf{K}]\{\mathbf{u}(t)\}=\{\mathbf{r}(t)\} \tag{1}$$

with initial conditions

$$\{\mathbf{u}(0)\}=\{\mathbf{u}_0\},\qquad \{\dot{\mathbf{u}}(0)\}=\{\mathbf{v}_0\} \tag{2}$$

where $[\mathbf{M}]$, $[\mathbf{C}]$, and $[\mathbf{K}]$ are time-invariant mass, damping, and stiffness matrices, respectively. The time-dependent displacement and velocity vectors are $\{\mathbf{u}(t)\}$ and $\{\mathbf{v}(t)\}=(\mathrm{d}/\mathrm{d}t)\{\mathbf{u}(t)\}=\{\dot{\mathbf{u}}(t)\}$, respectively. The known external excitation vector is $\{\mathbf{r}(t)\}$.

In 1994, Zhong and Williams [2] proposed a new precise time-step integration (PTI) algorithm. This method was shown to be able to give accurate solutions by using just one time step over a large interval with linearly varying excitations. A special feature in the precise time-step integration method is that the exponential matrix $\exp([\mathbf{w}]\cdot t)$ at $t=\Delta t$ is evaluated recursively as

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$$\exp([\mathbf{w}]\Delta t/2^k) = [\exp([\mathbf{w}]\Delta t/2^{k+1})]^2, \qquad k = m-1, \ldots, 2, 1, 0 \tag{3}$$

To initiate the evaluation, $\exp([\mathbf{w}]\Delta t/2^m)$ was evaluated from a truncated Taylor series. The number of recursive evaluations m was recommended to be 20 in the original paper [2].

The PTI method has been used successfully to evaluate structural responses under evolutionary random excitations [3], to solve asymmetric Riccati differential equations [4], and to study dynamic responses of a continuous beam under moving loads [5].

In 1995, Lin et al. [6] extended the PTI method to tackle sinusoidal loading and other more general loading forms by Fourier series. However, many terms in the Fourier series would be required for accurate solutions. Shen et al. [7] developed a parallel computing method to improve the efficiency of the PTI method. Fung [8] presented a PTI method by step-response and impulsive-response matrices to manipulate the second-order governing differential equations directly.

In all the aforementioned methods, the particular solutions arising from the excitation have to be computed separately. To avoid computing the particular solutions directly, Gu et al. [9] transformed the governing equations into an equivalent homogenous form by expanding the dimension of the matrices. However, the computational efficiency was found to be low, as the size of the computing matrices would increase by about 50%. Later, Wang et al. [10] expanded the dimension in a different way, in which the excitation would be separated into two parts, normally, a constant part and a time-varying part. The dimension of the expanded matrices was controlled by the time-varying part of the excitation. The increase of dimension was usually small and the computational effort would be lower than the computational effort required in computing the particular solution.

In [9,10], the governing equations were first transformed to first-order differential equations. It has been demonstrated [8] that by manipulating the second-order differential equations directly, the computational effort could be reduced by making using of symmetry and relations between the response matrices and their time derivatives. In this paper, the computational efficiency of the homogenized second-order differential equations is investigated. New response matrices corresponding to the Duhamel integral solutions are derived. The computational efficiency of the present methods is shown to be better than the methods presented in [9,10].

In Sec. II, the PTI method by step-response and impulsive-response matrices in [8] is reviewed briefly. Matrices relevant to the dimension expansion are given in Sec. III. In Secs. IV and V, two new PTI algorithms by response matrices are then proposed for excitations described by second-order and first-order differential equations, respectively. In Sec. VI, comparisons of the computational efforts between the existing PTI methods (with and without expanded dimension) and the two new PTI methods are presented. In Sec. VII, two numerical examples are used to illustrate the efficiency of the present new algorithms. A special computational procedure for periodic excitations is also discussed. Conclusions are then given in Sec. VIII.

II. PTI Method by Step-Response and Impulsive-Response Matrices

The transient displacement and velocity responses [8] of Eq. (1) can be written as

$$\{\mathbf{u}(t)\} = [\mathbf{G}(t)]\{\mathbf{u}_0 - \mathbf{u}_s(0)\} + [\mathbf{H}(t)]\{\mathbf{v}_0 - \mathbf{v}_s(0)\} + \{\mathbf{u}_s(t)\} \tag{4}$$

$$\{\mathbf{v}(t)\} = [\dot{\mathbf{G}}(t)]\{\mathbf{u}_0 - \mathbf{u}_s(0)\} + [\dot{\mathbf{H}}(t)]\{\mathbf{v}_0 - \mathbf{v}_s(0)\} + \{\mathbf{v}_s(t)\} \tag{5}$$

where $\{\mathbf{u}_s(t)\}$ is the steady-state response corresponding to $\{\mathbf{r}(t)\}$, $\{\mathbf{v}_s(t)\} = \{\dot{\mathbf{u}}_s(t)\}$, $\{\mathbf{u}_0\}$, and $\{\mathbf{v}_0\}$ are the given initial conditions, $[\mathbf{G}(t)]$ and $[\mathbf{H}(t)]$ are the step-response and impulsive-response matrices, respectively.

A. Step-Response and Impulsive-Response Matrices

For a given $[\mathbf{M}]$, $[\mathbf{C}]$, $[\mathbf{K}]$, Δt, and m, the step-response matrix $[\mathbf{G}(t)]$ and the impulsive-response matrix $[\mathbf{H}(t)]$, and their derivatives at $t = \Delta t$, can be computed [8] as follows:

1) Compute the initial matrices $[\mathbf{H}(\Delta t/2^m)]$ and $[\mathbf{J}(\Delta t/2^m)] = [\dot{\mathbf{H}}(\Delta t/2^m)]$ from

$$\left[\mathbf{H}\left(\frac{\Delta t}{2^m}\right)\right] = [\dot{\mathbf{H}}(0)]\cdot\frac{\Delta t}{2^m} + [\ddot{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^2}{2!} + [\dddot{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^3}{3!} + \cdots + [\overset{(n)}{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^n}{n!} + \cdots \tag{6a}$$

$$\left[\mathbf{J}\left(\frac{\Delta t}{2^m}\right)\right] = \left[\dot{\mathbf{H}}\left(\frac{\Delta t}{2^m}\right)\right] = [\dot{\mathbf{H}}(0)] + [\ddot{\mathbf{H}}(0)]\cdot\left(\frac{\Delta t}{2^m}\right) + [\dddot{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^2}{2!} + \cdots + [\overset{(n)}{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^{(n-1)}}{(n-1)!} + \cdots \tag{6b}$$

where

$$[\dot{\mathbf{H}}(0)] = [\mathbf{I}], \qquad [\ddot{\mathbf{H}}(0)] = -[\mathbf{M}]^{-1}[\mathbf{C}]$$
$$[\overset{(i+2)}{\mathbf{H}}(0)] = -[\overset{(i)}{\mathbf{H}}(0)][\mathbf{M}]^{-1}[\mathbf{K}] - [\overset{(i+1)}{\mathbf{H}}(0)][\mathbf{M}]^{-1}[\mathbf{C}] \tag{7}$$
$$i = 1, 2, 3, \ldots$$

To reduce the truncation error, an auxiliary matrix $[\mathbf{J}_a(\Delta t/2^m)] = [\mathbf{J}(\Delta t/2^m)] - [\mathbf{I}]$ is computed instead of $[\mathbf{J}(\Delta t/2^m)]$ here. In other words,

$$\left[\mathbf{J}_a\left(\frac{\Delta t}{2^m}\right)\right] = \left[\mathbf{J}\left(\frac{\Delta t}{2^m}\right)\right] - [\mathbf{I}] = [\ddot{\mathbf{H}}(0)]\cdot\left(\frac{\Delta t}{2^m}\right) + [\dddot{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^2}{2!} + \cdots + [\overset{(n)}{\mathbf{H}}(0)]\cdot\frac{(\Delta t/2^m)^{(n-1)}}{(n-1)!} + \cdots \tag{8}$$

2) Compute $[\mathbf{H}(\Delta t)]$ and $[\mathbf{J}_a(\Delta t)] = [\mathbf{J}(\Delta t)] - [\mathbf{I}]$ recursively from

$$\begin{aligned}[\mathbf{H}(\Delta t/2^{k-1})] &= [\mathbf{G}(\Delta t/2^k)][\mathbf{H}(\Delta t/2^k)] + [\mathbf{H}(\Delta t/2^k][\mathbf{J}(\Delta t/2^k)])\\ &= ([\mathbf{J}_a(\Delta t/2^k)] + [\mathbf{H}(\Delta t/2^k)][\mathbf{M}]^{-1}[\mathbf{C}])[\mathbf{H}(\Delta t/2^k)]\\ &\quad + [\mathbf{H}(\Delta t/2^k][\mathbf{J}_a(\Delta t/2^k)] + 2[\mathbf{H}(\Delta t/2^k)]\end{aligned} \tag{9a}$$

$$\begin{aligned}[\mathbf{J}_a[\Delta t/2^{k-1})] &= [\mathbf{J}_a(\Delta t/2^k)]^2 + 2[\mathbf{J}_a(\Delta t/2^k)]\\ &\quad - [\mathbf{H}(\Delta t/2^k)][\mathbf{M}]^{-1}[\mathbf{K}][\mathbf{H}(\Delta t/2^k)]\end{aligned} \tag{9b}$$

where $k = m, \ldots, 1$

3) Compute $[\dot{\mathbf{H}}(\Delta t)][\mathbf{G}(\Delta t)]$ and $[\dot{\mathbf{G}}(\Delta t)]$ from

$$[\dot{\mathbf{H}}(\Delta t)] = [\mathbf{J}(\Delta t)] = [\mathbf{J}_a(\Delta t)] + [\mathbf{I}] \tag{10a}$$

$$[\mathbf{G}(\Delta t)] = [\mathbf{J}_a(\Delta t)] + [\mathbf{I}] + [\mathbf{H}(\Delta t)][\mathbf{M}]^{-1}[\mathbf{C}] \tag{10b}$$

$$[\dot{\mathbf{G}}(\Delta t)] = [\mathbf{H}(\Delta t)][\mathbf{M}]^{-1}[\mathbf{K}] \tag{10c}$$

B. Steady-State Response

The computation of the steady-state response $\{\mathbf{u}_s(t)\}$ depends on the form of the excitation [11,12]. For example, if the excitation is given by

$$\{\mathbf{r}(t)\} = e^{\alpha t}\cdot\sum_{j=0}^{q}\{\mathbf{f}_j\}t^j \tag{11}$$

then it can be shown that $\{\mathbf{u}_s(t)\}$ could be written as [13]

$$\{\mathbf{u}_s(t)\} = e^{\alpha t} \cdot \sum_{j=0}^{q} \{\mathbf{u}_j\} t^j \tag{12}$$

where

$$\begin{aligned} \{\mathbf{u}_j\} = (\alpha^2[\mathbf{M}] + \alpha[\mathbf{C}] + [\mathbf{K}])^{-1} \cdot (\{\mathbf{f}_j\} - j([\mathbf{C}] \\ + 2\alpha[\mathbf{M}])\{\mathbf{u}_{j+1}\} - j(j+1)\{\mathbf{u}_{j+2}\}) \end{aligned} \tag{13}$$

$$j = 0, 1, \ldots, q \quad \text{with} \quad \{\mathbf{u}_{q+1}\} = \{\mathbf{u}_{q+2}\} = \{\mathbf{0}\}$$

Also,

$$\{\mathbf{v}_s(t)\} = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{u}_s(t)\} = \alpha \cdot \{\mathbf{u}_s(t)\} + e^{\alpha t} \cdot \sum_{j=1}^{q} j\{\mathbf{u}_j\} t^{j-1} \tag{14}$$

III. Dimension Expansion

To avoid computing the particular solutions, Gu et al. [9] and Wang et al. [10] proposed expanding the dimension of the system after transforming the second-order nonhomogeneous equations into some equivalent first-order homogeneous equations.

In this paper, the second-order equations are manipulated directly without transforming to the equivalent first-order equations. Also, the excitations could be described by either first-order or second-order differential equations, i.e., $\{\mathbf{r}(t)\} = [\mathbf{f}]\{\mathbf{Z}(t)\}$ where $\{\mathbf{Z}(t)\}$ is the solution of a system of first-order equation $\{\dot{\mathbf{Z}}(t)\} = [\mathbf{S}_1]\{\mathbf{Z}(t)\}$ or second-order equation $\{\ddot{\mathbf{Z}}(t)\} = [\mathbf{S}_2]\{\mathbf{Z}(t)\}$.

In the following, excitations in the following form are considered:

$$\{\mathbf{r}(t)\} = e^{\alpha t} \cdot \sum_{i=1}^{g} [\{\mathbf{a}_i\} t^{i-1} \sin(c_i t) + \{\mathbf{b}_i\} t^{i-1} \cos(c_i t)] \tag{15}$$

Equation (15) treats many loading forms, for example, linear, polynomial, and Fourier loading forms as special cases. This form is rather general and could be used to approximate many different types of excitations. However, for those excitations that could not be described by first-order or second-order differential equations, the present method may not be applicable.

Let

$$z_i^s = e^{\alpha t} \cdot t^{i-1} \cdot \sin(c_i t) \quad \text{and}$$
$$z_i^c = e^{\alpha t} \cdot t^{i-1} \cdot \cos(c_i t), \qquad i = 1, 2, \ldots, g \tag{16}$$

Then,

$$\begin{gathered} \{\mathbf{r}(t)\} = [\mathbf{f}] \cdot \{\mathbf{Z}(t)\} \\ \{\dot{\mathbf{Z}}(t)\} = [\mathbf{S}_1]\{\mathbf{Z}(t)\}, \quad \text{and} \quad \{\ddot{\mathbf{Z}}(t)\} = [\mathbf{S}_2]\{\mathbf{Z}(t)\} \end{gathered} \tag{17}$$

where

$$[\mathbf{f}] = [\,\{\mathbf{a}_1\} \quad \{\mathbf{b}_1\} \quad \{\mathbf{a}_2\} \quad \{\mathbf{b}_2\} \quad \cdots \quad \{\mathbf{a}_g\} \quad \{\mathbf{b}_g\}\,] \tag{18}$$

$$\{\mathbf{Z}(t)\} = \left\{ z_1^s \quad z_1^c \quad z_2^s \quad z_2^c \quad \cdots \quad z_g^s \quad z_g^c \right\}^T \tag{19}$$

$$[\mathbf{S}_1] = \begin{bmatrix} \alpha & c_1 & & & & & & & & \\ -c_1 & \alpha & & & & & & & & \\ 1 & 0 & \alpha & c_2 & & & & & & \\ 0 & 1 & -c_2 & \alpha & 0 & & & & & \\ & & 2 & 0 & \alpha & c_3 & & & & \\ & & 0 & 2 & -c_3 & \alpha & \ddots & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & (g-1) & 0 & \alpha & c_g \\ & & & & & & 0 & (g-1) & -c_g & \alpha \end{bmatrix} \tag{20a}$$

$[\mathbf{S}_2] = [\mathbf{S}_1][\mathbf{S}_1] =$

$$\begin{bmatrix} \alpha^2-c_1^2 & 2\alpha c_1 & 0 & 0 & \cdots & & & & & & & \\ -2\alpha c_1 & \alpha^2-c_1^2 & 0 & 0 & & & & & & & & \\ 2\alpha & c_1+c_2 & \alpha^2-c_2^2 & 2\alpha c_2 & \ddots & & & & & & & \\ -(c_1+c_2) & 2\alpha & -2\alpha c_2 & \alpha^2-c_2^2 & 0 & & & & & & & \\ 1\cdot 2 & 0 & 2\cdot 2\alpha & 2(c_2+c_3) & \alpha^2-c_3^2 & 2\alpha c_3 & \ddots & & \ddots & & \ddots & \vdots \\ 0 & 1\cdot 2 & -2(c_2+c_3) & 2\cdot 2\alpha & -2\alpha c_3 & \alpha^2-c_3^2 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & (g-2)(g-1) & 0 & 2(g-1)\alpha & (g-1)(c_{g-1}+c_g) & \alpha^2-c_g^2 & (g-1)\alpha c_g \\ 0 & 0 & \cdots & 0 & 0 & 0 & (g-2)(g-1) & -(g-1)(c_{g-1}+c_g) & 2(g-1)\alpha & -(g-1)\alpha c_g & \alpha^2-c_g^2 \end{bmatrix} \tag{20b}$$

The explicit forms of matrices $[\mathbf{S}_1]$ and $[\mathbf{S}_2]$ for some typical excitations are given next:

1) Linearly varying excitations

$$\{\mathbf{r}(t)\} = \{\mathbf{r}_0\} + \{\mathbf{r}_1\} \cdot t \tag{21}$$

where

$$[\mathbf{f}] = [\,\mathbf{r}_0 \quad \mathbf{r}_1\,], \qquad \{\mathbf{Z}(t)\} = [\,1 \quad t\,]^T \tag{22}$$

$$[\mathbf{S}_1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad [\mathbf{S}_2] = [\mathbf{0}] \tag{23}$$

2) Polynomial excitations

$$\{\mathbf{r}(t)\} = \{\mathbf{r}_0\} + \{\mathbf{r}_1\}t + \cdots + \{\mathbf{r}_{g-2}\}t^{g-2} + \{\mathbf{r}_{g-1}\}t^{g-1} \tag{24}$$

where

$$\begin{aligned} \{\mathbf{f}\} &= \{\, \mathbf{r}_0 \quad \mathbf{r}_1 \quad \cdots \quad \mathbf{r}_{g-1} \,\} \\ \{\mathbf{Z}(t)\} &= [\, 1 \quad t \quad t^2 \quad \cdots \quad t^{g-1} \,]^T \end{aligned} \tag{25}$$

$$[\mathbf{S}_1] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & g-1 & 0 \end{bmatrix}$$

$$[\mathbf{S}_2] = \begin{bmatrix} 0 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & & 0 & 0 \\ 2\cdot 1 & 0 & 0 & & \vdots & \vdots \\ 0 & 3\cdot 2 & 0 & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & (g-1)\cdot(g-2) & 0 & 0 \end{bmatrix} \tag{26}$$

3) Fourier representations

$$\{\mathbf{r}(t)\} = \{\mathbf{r}_0\} + \sum_{i=1}^{g} \left(\left\{\mathbf{r}_i^c\right\} \cos(c_i t) + \left\{\mathbf{r}_i^s\right\} \sin\left(c_i^t\right) \right) \tag{27}$$

where

$$[\mathbf{f}] = \left[\{\mathbf{r}_0\} \quad \left\{\mathbf{r}_1^c\right\} \quad \left\{\mathbf{r}_1^s\right\} \quad \left\{\mathbf{r}_2^c\right\} \quad \left\{\mathbf{r}_2^s\right\} \quad \cdots \quad \left\{\mathbf{r}_g^c\right\} \quad \left\{\mathbf{r}_g^s\right\} \right] \tag{28}$$

$$\{\mathbf{Z}(t)\} = \{\, 1 \quad \cos(c_1 t) \quad \sin(c_1 t) \quad \cos(c_2 t) \quad \sin(c_2 t) \quad \cdots \quad \cos(c_g t) \quad \sin(c_g t) \,\}^T \tag{29}$$

$$[\mathbf{S}_1] = \begin{bmatrix} 0 & 0 & & & & & & \\ 0 & 0 & -c_1 & & & & & \\ & c_1 & 0 & 0 & & & & \\ & & 0 & 0 & -c_2 & & & \\ & & & c_2 & 0 & & & \\ & & & & & \ddots & \ddots & \\ & & & & & \ddots & 0 & -c_g \\ & & & & & & c_g & 0 \end{bmatrix} \tag{30a}$$

$$[\mathbf{S}_2] = \begin{bmatrix} 0 & 0 & & & & & \\ 0 & -c_1^2 & 0 & & & & \\ & 0 & -c_1^2 & & & & \\ & & & -c_2^2 & \ddots & & \\ & & & \ddots & -c_2^2 & & \\ & & & & & \ddots & \\ & & & & & -c_g^2 & 0 \\ & & & & & 0 & -c_g^2 \end{bmatrix} \tag{30b}$$

Note that if a time shift is applied to the excitation, i.e.,

$$\{\bar{\mathbf{r}}(t)\} = \{\mathbf{r}(t - t^*)\} \tag{31}$$

then, $\{\bar{\mathbf{r}}(t)\} = [\mathbf{f}]\{\bar{\mathbf{Z}}\}$, where $\{\bar{\mathbf{Z}}(t)\} = \{\mathbf{Z}(t - t^*)\}$ with $\{\bar{\mathbf{Z}}\}$ governed by the same $[\mathbf{S}_1]$ and $[\mathbf{S}_2]$.

When the excitation is described by a second-order differential equation $\{\mathbf{r}(t)\} = [\mathbf{f}]\{\mathbf{Z}\}$ with $\{\ddot{\mathbf{Z}}\} = [\mathbf{S}_2]\{\mathbf{Z}\}$, Eq. (1) can be expanded into a homogeneous form as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}(t) \\ \ddot{\mathbf{Z}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}}(t) \\ \dot{\mathbf{Z}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{f} \\ \mathbf{0} & -\mathbf{S}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{Z}(t) \end{Bmatrix} = \{\mathbf{0}\} \tag{32}$$

Alternatively, if the excitation is described by a first-order differential equation, i.e., $\{\mathbf{r}(t)\} = [\mathbf{f}]\{\mathbf{Z}\}$ with $\{\dot{\mathbf{Z}}(t)\} = [\mathbf{S}_1]\{\mathbf{Z}(t)\}$, then Eq. (1) becomes

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}(t) \\ \ddot{\mathbf{Z}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}}(t) \\ \dot{\mathbf{Z}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{f} \\ \mathbf{0} & -\mathbf{S}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{Z}(t) \end{Bmatrix} = \{\mathbf{0}\} \tag{33}$$

It should be noted that although Eq. (32) can be tackled by many established algorithms, Eq. (33) may impose some difficulties, as the first matrix is singular. A more detailed discussion on how to solve the two equations is given in the next two sections.

IV. Excitations Described by Second-Order Differential Equations

The PTI method by step-response and impulsive-response matrices [8] can be used to solve the homogeneous second-order differential equation in Eq. (32) directly, without the need to compute any steady-state response. Equation (32) can be expressed as

$$[\mathbf{M}^*]\{\ddot{\mathbf{u}}^*(t)\} + [\mathbf{C}^*]\{\dot{\mathbf{u}}^*(t)\} + [\mathbf{K}^*]\{\mathbf{u}^*(t)\} = 0 \tag{34}$$

where

$$\{\mathbf{u}^*(t)\} = \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{Z}(t) \end{Bmatrix}, \qquad [\mathbf{M}^*] = \begin{bmatrix} \mathbf{M} & \\ & \mathbf{I} \end{bmatrix}$$

$$[\mathbf{C}^*] = \begin{bmatrix} \mathbf{C} & \\ & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad [\mathbf{K}^*] = \begin{bmatrix} \mathbf{K} & -\mathbf{f} \\ \mathbf{0} & -\mathbf{S}_2 \end{bmatrix}$$

As described in Sec. II, the solution at time $t = \Delta t$ of Eq. (34) can be obtained as

$$\begin{Bmatrix} \mathbf{u}^*(\Delta t) \\ \mathbf{v}^*(\Delta t) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^*(\Delta t) & \mathbf{H}^*(\Delta t) \\ \dot{\mathbf{G}}^*(\Delta t) & \dot{\mathbf{H}}^*(\Delta t) \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{u}_0^* \\ \mathbf{v}_0^* \end{Bmatrix}$$

$$\text{with } \{\mathbf{u}_0^*\} = \begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{Z}(0) \end{Bmatrix} \quad \text{and} \quad \left\{\mathbf{v}_0^*\right\} = \begin{Bmatrix} \dot{\mathbf{u}}(0) \\ \dot{\mathbf{Z}}(0) \end{Bmatrix} \tag{35}$$

For one time step Δt, the initial recurrence formulas of $[\mathbf{H}^*(\Delta t/2^m)]$ and $[\mathbf{J}_a^*(\Delta t/2^m)] = [\mathbf{J}^*(\Delta t/2^m)] - [\mathbf{I}]$ can be computed from the Taylor series. The recurrence formulas of the matrices $[\mathbf{G}^*(\Delta t)]$, $[\mathbf{H}^*(\Delta t)]$, and their derivatives can be evaluated by replacing the $[\mathbf{M}]$, $[\mathbf{C}]$, and $[\mathbf{K}]$ by $[\mathbf{M}^*]$, $[\mathbf{C}^*]$, and $[\mathbf{K}^*]$ in Eqs. (6) and

(8–10). A four-term truncated Taylor series approximation recommended in [8] could be used. The recursive computational procedure is then carried out m times to get $[\mathbf{H}^*(\Delta t)]$ and $[\mathbf{J}^*(\Delta t)]$. A disadvantage of this algorithm is that $[\mathbf{K}^*]$ is not symmetrical. Hence, the computational effort could not be optimized further by making use of any symmetry property.

V. Excitations Described by First-Order Differential Equations

As mentioned in Sec. III, Eq. (33) cannot be tackled directly. However, it can be shown that the solution of Eq. (33) can be expressed in terms of the given initial conditions $\{\mathbf{u}_0\}$, $\{\mathbf{v}_0\}$, and $\{\mathbf{Z}_0\}$. In other words, the displacement response $\{\mathbf{u}(t)\}$ and the velocity response $\{\mathbf{v}(t)\}$ can be expressed as

$$\{\mathbf{u}(t)\} = [\mathbf{G}(t)]\{\mathbf{u}_0\} + [\mathbf{H}(t)]\{\mathbf{v}_0\} + [\mathbf{D}(t)]\{\mathbf{Z}_0\} \tag{36a}$$

$$\{\mathbf{v}(t)\} = [\dot{\mathbf{G}}(t)]\{\mathbf{u}_0\} + [\dot{\mathbf{H}}(t)]\{\mathbf{v}_0\} + [\dot{\mathbf{D}}(t)]\{\mathbf{Z}_0\} \tag{36b}$$

It should be noted that the solution in Eq. (1) or Eq. (33) can also be written as

$$\{\mathbf{u}(t)\} = [\mathbf{G}(t)]\{\mathbf{u}_0\} + [\mathbf{H}(t)]\{\mathbf{v}_0\} + \{\mathbf{u}_d(t)\} \tag{37}$$

where $\{\mathbf{u}_d(t)\}$ is a particular solution given by the Duhamel integral with zero initial conditions.

Comparing Eqs. (36a) and (37), it reveals that

$$\{\mathbf{u}_d(t)\} = [\mathbf{D}(t)]\{\mathbf{Z}_0\} \tag{38}$$

As a result, the particular solution corresponding to the Duhamel integral can be expressed in terms of $\{\mathbf{Z}_0\}$ and a Duhamel-response matrix $[\mathbf{D}(t)]$.

The computation of $[\mathbf{H}(t)]$ and $[\mathbf{G}(t)]$ (or $[\mathbf{J}(t)]$) has been discussed in [8] and Sec. II of this paper. In the following, the computation of $[\mathbf{D}(t)]$ is considered.

Note that $[\mathbf{D}(t)]$ satisfies the following equation:

$$[\mathbf{M}][\ddot{\mathbf{D}}(t)]\{\mathbf{Z}_0\} + [\mathbf{C}][\dot{\mathbf{D}}(t)]\{\mathbf{Z}_0\} + [\mathbf{K}][\mathbf{D}(t)]\{\mathbf{Z}_0\} = [\mathbf{f}]\{\mathbf{Z}(t)\} \tag{39}$$

with $[\mathbf{D}(0)] = [\dot{\mathbf{D}}(0)] = [\mathbf{0}]$, where $[\mathbf{0}]$ is the zero matrix, and $\{\dot{\mathbf{Z}}(t)\} = [\mathbf{S}_1]\{\mathbf{Z}(t)\}$.

The Taylor series of $[\mathbf{D}(t)]$ and $[\dot{\mathbf{D}}(t)]$ at $t = 0$ can be written as

$$[\mathbf{D}(t)] = [\ddot{\mathbf{D}}(0)]\cdot\frac{t^2}{2!} + [\overset{(3)}{\mathbf{D}}(0)]\cdot\frac{t^3}{3!} + [\overset{(4)}{\mathbf{D}}(0)]\cdot\frac{t^4}{4!} + \cdots[\overset{(n+1)}{\mathbf{D}}(0)]\cdot\frac{t^{n+1}}{(n+1)!} + \cdots \tag{40a}$$

$$[\dot{\mathbf{D}}(t)] = [\ddot{\mathbf{D}}(0)]\cdot t + [\overset{(3)}{\mathbf{D}}(0)]\cdot\frac{t^2}{2!} + [\overset{(4)}{\mathbf{D}}(0)]\cdot\frac{t^3}{3!} + \cdots + [\overset{(n+1)}{\mathbf{D}}(0)]\cdot\frac{t^n}{n!} + \cdots \tag{40b}$$

Differentiating Eq. (39) with respect to t and set $t = 0$ repeatedly, one has

$$[\mathbf{M}][\ddot{\mathbf{D}}(0)]\{\mathbf{Z}_0\} + [\mathbf{C}][\dot{\mathbf{D}}(0)]\{\mathbf{Z}_0\} + [\mathbf{K}][\mathbf{D}(0)]\{\mathbf{Z}_0\} = [\mathbf{f}]\{\mathbf{Z}_0\} \tag{41}$$

$$[\mathbf{M}][\overset{(3)}{\mathbf{D}}(0)]\{\mathbf{Z}_0\} + [\mathbf{C}][\ddot{\mathbf{D}}(0)]\{\mathbf{Z}_0\} + [\mathbf{K}][\dot{\mathbf{D}}(0)]\{\mathbf{Z}_0\} = [\mathbf{f}]\{\dot{\mathbf{Z}}(0)\} = [\mathbf{f}][\mathbf{S}_1]\{\mathbf{Z}_0\} \tag{42}$$

$$\vdots$$

$$[\mathbf{M}][\overset{(i+1)}{\mathbf{D}}(0)]\{\mathbf{Z}_0\} + [\mathbf{C}][\overset{(i)}{\mathbf{D}}(0)]\{\mathbf{Z}_0\} + [\mathbf{K}][\overset{(i-1)}{\mathbf{D}}(0)]\{\mathbf{Z}_0\} = [\mathbf{f}]\{\overset{(i-1)}{\mathbf{Z}}(0)\} = [\mathbf{f}][\mathbf{S}_1]^{i-1}\{\mathbf{Z}_0\} \tag{43}$$

Making use of Eq. (7) and, after some algebraic manipulations, $[\overset{(i)}{\mathbf{D}}(0)]$ can be expressed as

$$[\ddot{\mathbf{D}}(0)] = [\mathbf{M}]^{-1}[\mathbf{f}], \qquad [\overset{(3)}{\mathbf{D}}(0)] = [\ddot{\mathbf{H}}(0)][\mathbf{M}]^{-1}[\mathbf{f}] + [\ddot{\mathbf{D}}(0)][\mathbf{S}_1] \tag{44}$$

$$[\overset{(i+1)}{\mathbf{D}}(0)] = [\overset{(i)}{\mathbf{H}}(0)][\mathbf{M}]^{-1}[\mathbf{f}] + [\overset{(i)}{\mathbf{D}}(0)][\mathbf{S}_1], \qquad i = 2,3,4,\ldots \tag{45}$$

Hence, the matrix $[\mathbf{D}(t)]$ can be evaluated efficiently.

Similar to $[\mathbf{H}(t)]$ and $[\mathbf{G}(t)]$, using the Taylor series to compute $[\mathbf{D}(t)]$ at $t = \Delta t$ would require Δt to be very small in general. In the following, the squaring and scaling technique is used to compute $[\mathbf{D}(t)]$ recursively.

A. Computation of $[\mathbf{E}(t)]$, $[\mathbf{D}(t)]$, and $[\dot{\mathbf{D}}(t)]$

From Eqs. (36a) and (36b), the solution of Eq. (1) at $t = \Delta t$ can be expressed as

$$\begin{Bmatrix} \mathbf{u}(\Delta t) \\ \mathbf{v}(\Delta t) \\ \mathbf{Z}(\Delta t) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}(\Delta t) & \mathbf{H}(\Delta t) & \mathbf{D}(\Delta t) \\ \dot{\mathbf{G}}(\Delta t) & \dot{\mathbf{H}}(\Delta t) & \dot{\mathbf{D}}(\Delta t) \\ 0 & 0 & \mathbf{E}(\Delta t) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \\ \mathbf{Z}_0 \end{Bmatrix} \tag{46}$$

where $[\mathbf{E}(\Delta t)] = \exp([\mathbf{S}_1]\Delta t)$, so that $\{\mathbf{Z}(\Delta t)\} = [\mathbf{E}(\Delta t)]\cdot\{\mathbf{Z}_0\}$ is the solution of $\{\dot{\mathbf{Z}}(t)\} = [\mathbf{S}_1]\cdot\{\mathbf{Z}(t)\}$ at $t = \Delta t$. The matrix $[\mathbf{E}(\Delta t)]$ also can be obtained by using Taylor series.

$$[\mathbf{E}(\Delta t)] = \sum_{\mathbf{i=0}}^{\infty}[\mathbf{S}_1]^i\cdot\frac{\Delta t^i}{i!} = [\mathbf{I}] + [\mathbf{S}_1]\cdot\Delta t + [\mathbf{S}_1]^2\cdot\frac{\Delta t^2}{2!} + [\mathbf{S}_1]^3\cdot\frac{\Delta t^3}{3!} + \cdots \tag{47}$$

or by using the squaring and scaling technique as

$$[\mathbf{E}(\Delta t)] = \left[\mathbf{E}\left(\frac{\Delta t}{2^m}\right)\right]^{2^m} = ([\mathbf{I}] + [\mathbf{E}_a])^{2^m} \tag{48}$$

where

$$[\mathbf{E}_a] = \sum_{i=1}^{\infty}\frac{[\mathbf{S}_1]^i}{i!}\left(\frac{\Delta t}{2^m}\right)^i$$

Obviously, the same solution can be obtained by applying a time step $\Delta t/2$ first and then followed by another $\Delta t/2$. In other words,

$$\begin{Bmatrix} \mathbf{u}(\Delta t) \\ \mathbf{v}(\Delta t) \\ \mathbf{Z}(\Delta t) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}(\Delta t/2) & \mathbf{H}(\Delta t/2) & \mathbf{D}(\Delta t/2) \\ \dot{\mathbf{G}}(\Delta t/2) & \dot{\mathbf{H}}(\Delta t/2) & \dot{\mathbf{D}}(\Delta t/2) \\ 0 & 0 & \mathbf{E}(\Delta t/2) \end{bmatrix} \begin{Bmatrix} \mathbf{u}(\Delta t/2) \\ \mathbf{v}(\Delta t/2) \\ \mathbf{Z}(\Delta t/2) \end{Bmatrix} \tag{49a}$$

$$= \begin{bmatrix} \mathbf{G}(\Delta t/2) & \mathbf{H}(\Delta t/2) & \mathbf{D}(\Delta t/2) \\ \dot{\mathbf{G}}(\Delta t/2) & \dot{\mathbf{H}}(\Delta t/2) & \dot{\mathbf{D}}(\Delta t/2) \\ 0 & 0 & \mathbf{E}(\Delta t/2) \end{bmatrix} \begin{bmatrix} \mathbf{G}(\Delta t/2) & \mathbf{H}(\Delta t/2) & \mathbf{D}(\Delta t/2) \\ \dot{\mathbf{G}}(\Delta t/2) & \dot{\mathbf{H}}(\Delta t/2) & \dot{\mathbf{D}}(\Delta t/2) \\ 0 & 0 & \mathbf{E}(\Delta t/2) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \\ \mathbf{Z}_0 \end{Bmatrix} \tag{49b}$$

Comparing Eqs. (46) and (49b), it can be seen that the recurrence equations for $[\mathbf{E}(\Delta t)]$, $[\mathbf{D}(\Delta t)]$, and $[\dot{\mathbf{D}}(\Delta t)]$ are

$$[\mathbf{E}(\Delta t)] = [\mathbf{E}(\Delta t/2)][\mathbf{E}(\Delta t/2)] \tag{50a}$$

$$[\mathbf{D}(\Delta t)] = [\mathbf{G}(\Delta t/2)][\mathbf{D}(\Delta t/2)] + [\mathbf{H}(\Delta t/2)][\dot{\mathbf{D}}(\Delta t/2)] + [\mathbf{D}(\Delta t/2)][\mathbf{E}(\Delta t/2)] \tag{50b}$$

$$[\dot{\mathbf{D}}(\Delta t)] = [\dot{\mathbf{G}}(\Delta t/2)][\mathbf{D}(\Delta t/2)] + [\dot{\mathbf{H}}(\Delta t/2)][\dot{\mathbf{D}}(\Delta t/2)] + [\dot{\mathbf{D}}(\Delta t/2)][\mathbf{E}(\Delta t/2)] \tag{50c}$$

B. Relation of $[\mathbf{D}(t)]$ and $[\dot{\mathbf{D}}(t)]$

The present precise time-step integration method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension requires the computation of matrices $[\mathbf{D}(\Delta t)]$ and $[\dot{\mathbf{D}}(\Delta t)]$. This is a drawback of the present method and it is not efficient to compute the Duhamel-response matrix $[\mathbf{D}(t)]$ and its derivative $[\dot{\mathbf{D}}(t)]$ separately. To reduce the computational cost, the relationship between the Duhamel-response matrix $[\mathbf{D}(\Delta t)]$ and its derivative $[\dot{\mathbf{D}}(\Delta t)]$ is established next.

Using the Duhamel integral, the particular solution can be computed from

$$\{\mathbf{u}_d(t)\} = [\mathbf{D}(t)]\cdot\{\mathbf{Z}_0\} = \int_0^t [\mathbf{H}(s)]\cdot\{\mathbf{R}(t-s)\}\,\mathrm{d}s \tag{51}$$

where $\{\mathbf{R}(t)\} = [\mathbf{M}]^{-1}[\mathbf{f}]\{\mathbf{Z}(t)\}$ and $\{\mathbf{Z}(t)\} = [\mathbf{E}(t)]\{\mathbf{Z}_0\} = \exp([\mathbf{S}_1]\cdot t)\cdot\{\mathbf{Z}_0\}$

Hence,

$$[\mathbf{D}(t)]\cdot\{\mathbf{Z}_0\} = \int_0^t [\mathbf{H}(s)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}]\cdot\exp([\mathbf{S}_1](t-s))\cdot\{\mathbf{Z}_0\}\,\mathrm{d}s \tag{52}$$

Differentiate Eq. (52) with respect to t,

$$\begin{aligned}[\dot{\mathbf{D}}(t)]\cdot\{\mathbf{Z}_0\} &= ([\mathbf{H}(s)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}]\cdot\exp([\mathbf{S}_1](t-s))\{\mathbf{Z}_0\}\Big|_{s=t} \\ &\quad + \int_0^t [\mathbf{H}(s)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}]\cdot\exp([\mathbf{S}_1](t-s))\,\mathrm{d}s\cdot[\mathbf{S}_1]\{\mathbf{Z}_0\} \\ &= [\mathbf{H}(t)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}]\cdot\{\mathbf{Z}_0\} + \int_0^t [\mathbf{H}(s)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}] \\ &\quad \cdot\exp([\mathbf{S}_1](t-s))\,\mathrm{d}s\cdot[\mathbf{S}_1]\cdot\{\mathbf{Z}_0\}\{\mathbf{Z}_0\}\end{aligned} \tag{53}$$

Making use of Eq. (51),

$$[\dot{\mathbf{D}}(t)]\cdot\{\mathbf{Z}_0\} = [\mathbf{H}(t)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}]\cdot\{\mathbf{Z}_0\} + [\mathbf{D}(t)]\cdot[\mathbf{S}_1]\cdot\{\mathbf{Z}_0\} \tag{54}$$

As a result, $[\dot{\mathbf{D}}(t)]$ can be obtained through Duhamel-response matrix $[\mathbf{D}(t)]$ and impulsive-response matrix $[\mathbf{H}(t)]$ as follows:

$$[\dot{\mathbf{D}}(t)] = [\mathbf{H}(t)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}] + [\mathbf{D}(t)]\cdot[\mathbf{S}_1] \tag{55}$$

Equation (55) could also be verified by comparing the Taylor series of $[\mathbf{D}(t)]$ and $[\dot{\mathbf{D}}(t)]$.

In the present algorithm, given $[\mathbf{H}(\Delta t/2)]$, $[\mathbf{J}(\Delta t/2)]$, and $[\mathbf{D}(\Delta t/2)]$, the matrices $[\mathbf{E}(\Delta t)]$ and $[\mathbf{D}(\Delta t)]$ can be computed from Eqs. (50a) and (50b), respectively. The matrix $[\dot{\mathbf{D}}(\Delta t)]$ in Eq. (50c) can be obtained from Eq. (55). Equal order truncated Taylor series approximations can be used for $[\mathbf{H}(\Delta t/2^m)]$, $[\mathbf{J}(\Delta t/2^m)]$, $[\mathbf{D}(\Delta t/2^m)]$, and $[\mathbf{E}(\Delta t/2^m)]$ initially. To reduce the truncation error, the auxiliary matrices $[\mathbf{J}_a(\Delta t)] = [\mathbf{J}(\Delta t/2)] - [\mathbf{I}]$ and $[\mathbf{E}_a(\Delta t)] = [\mathbf{E}(\Delta t/2)] - [\mathbf{I}]$ are computed. The computing procedure of this new algorithm is summarized in Table 1.

Note that the Duhamel-response matrix $[\mathbf{D}(\Delta t)]$, in general, is not a square matrix (and hence is not symmetrical normally). However, the computation of $[\mathbf{H}(\Delta t)]$ and $[\mathbf{J}(\Delta t)]$ can still make use of symmetry [8] to reduce the computational cost. Furthermore, the computation of the additional matrix $[\mathbf{E}(\Delta t)]$ only requires one matrix multiplication in each recursive evaluation.

In the present two new methods, the desirable time-step size Δt can be chosen independently of the highest frequency in the model. After the response matrices are evaluated, the time-step integration can be carried out with the time-step size chosen. The proposed methods are therefore very flexible. The time-step size need not be small and can be several times the longest period in the system.

VI. Computational Effort

In this section, the computational effort of the PTI method with expanded dimension given in [10], the PTI method by step-response and impulsive-response matrices given in [8], and the two new proposed methods are studied and compared.

Let N_s be the dimension of the system and g be the dimension of the vector $\{\mathbf{Z}(t)\}$. The solutions by different algorithms are given as follows:

1) The PTI method with expanded dimension [10]

$$\{\mathbf{U}^*(\Delta t)\} = \exp([\mathbf{H}^*]\cdot\Delta t)_{(2N_s+g)\times(2N_s+g)}\cdot\{\mathbf{U}_0^*\} \tag{56a}$$

2) The PTI method by step-response and impulsive-response matrices [8]

Table 1 Procedure of PTI method with step-response, impulsive-response, and Duhamel-response matrices

1) Compute the initial condition $[\mathbf{H}(\Delta t/2^m)]$, $[\mathbf{D}(\Delta t/2^m)]$, $[\mathbf{J}_a(\Delta t/2^m)] = [\mathbf{D}(\Delta t/2^m)] - [\mathbf{I}]$, and $[\mathbf{E}_a(\Delta t/2^m)] = [\mathbf{E}(\Delta t/2^m)] - [\mathbf{I}]$ from

$$[\mathbf{H}(\tfrac{\Delta t}{2^m})] = [\mathbf{I}]\cdot(\tfrac{\Delta t}{2^m}) - [\mathbf{M}]^{-1}[\mathbf{C}]\cdot\tfrac{(\Delta t/2^m)^2}{2!} + \{([\mathbf{M}]^{-1}[\mathbf{C}])^2 - [\mathbf{M}]^{-1}[\mathbf{K}]\}\cdot\tfrac{(\Delta t/2^m)^2}{3!} + \{-([\mathbf{M}]^{-1}[\mathbf{C}])^3 + [\mathbf{M}]^{-1}[\mathbf{C}][\mathbf{M}]^{-1}[\mathbf{K}] + [\mathbf{M}]^{-1}[\mathbf{K}][\mathbf{M}]^{-1}[\mathbf{C}]\}\cdot\tfrac{(\Delta t/2^m)^4}{4!}$$

$$[\mathbf{J}_a(\tfrac{\Delta t}{2^m})] = [\mathbf{J}](\tfrac{\Delta t}{2^m}) - [\mathbf{I}] = -[\mathbf{M}]^{-1}[\mathbf{C}]\cdot\tfrac{\Delta t}{2^m} + \{([\mathbf{M}]^{-1}[\mathbf{C}])^2 - [\mathbf{M}]^{-1}[\mathbf{K}]\}\cdot\tfrac{(\Delta t/2^m)^2}{2!} + \{-([\mathbf{M}]^{-1}[\mathbf{C}])^3 + [\mathbf{M}]^{-1}[\mathbf{C}][\mathbf{M}]^{-1}[\mathbf{K}] + [\mathbf{M}]^{-1}[\mathbf{K}][\mathbf{M}]^{-1}[\mathbf{C}]\}\cdot\tfrac{(\Delta t/2^m)^3}{3!}$$

$$[\mathbf{D}(\tfrac{\Delta t}{2^m})] = [\mathbf{M}]^{-1}[\mathbf{f}]\cdot\tfrac{(\Delta t/2^m)^2}{2!} + (-[\mathbf{M}]^{-1}[\mathbf{C}][\mathbf{M}]^{-1}[\mathbf{f}] + [\mathbf{M}]^{-1}[\mathbf{f}][\mathbf{S}_1])\cdot\tfrac{(\Delta t/2^m)^3}{3!} + \{[([\mathbf{M}]^{-1}[\mathbf{C}])^2 - [\mathbf{M}]^{-1}[\mathbf{K}]]\cdot[\mathbf{M}]^{-1}[\mathbf{f}] + ([\mathbf{M}]^{-1}[\mathbf{f}][\mathbf{S}_1] - [\mathbf{M}]^{-1}[\mathbf{C}][\mathbf{M}]^{-1}[\mathbf{f}])\cdot[\mathbf{S}_1]\}\cdot\tfrac{(\Delta t/2^m)^4}{4!}$$

$$[\mathbf{E}_a(\tfrac{\Delta t}{2^m})] = [\mathbf{E}(\tfrac{\Delta t}{2^m})] - [\mathbf{I}] = [\mathbf{S}_1]\cdot(\tfrac{\Delta t}{2^m}) + [\mathbf{S}_1]^2\cdot\tfrac{(\Delta t/2^m)^2}{2!} + [\mathbf{S}_1]^3\cdot\tfrac{(\Delta t/2^m)^3}{3!} + [\mathbf{S}_1]^4\cdot\tfrac{(\Delta t/2^m)^4}{4!}$$

2) Compute $[\mathbf{H}(\Delta t)]$, $[\mathbf{J}_a(\Delta t)]$, $[\mathbf{D}(\Delta t)]$, and $[\mathbf{E}_a(\Delta t)]$ recursively from

$$[\mathbf{H}(\Delta t/2^{k-1})] = [\mathbf{G}(\Delta t/2^k)][\mathbf{H}(\Delta t/2^k)] + [\mathbf{H}(\Delta t/2^k]([\mathbf{J}_a(\Delta t/2^k)] + [\mathbf{I}])$$

$$[\mathbf{J}_a(\Delta t/2^{k-1})] = [\mathbf{J}_a(\Delta t/2^k)]^2 + 2[\mathbf{J}(\Delta t/2^k)] - [\mathbf{H}(\Delta t/2^k][\mathbf{M}]^{-1}[\mathbf{K}][\mathbf{H}(\Delta t/2^k)]$$

$$[\mathbf{D}(\Delta t/2^{k-1})] = [\mathbf{G}(\Delta t/2^k)][\mathbf{D}(\Delta t/2^k)] + [\mathbf{D}(\Delta t/2^k)]\cdot([\mathbf{E}_a(\Delta t/2^k)] + [\mathbf{I}]) + [\mathbf{H}(\Delta t/2^k)]([\mathbf{H}(\Delta t/2^k)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}] + [\mathbf{D}(\Delta t/2^k)]\cdot[\mathbf{S}_1])$$

$$([\mathbf{E}_a(\Delta t/2^{k-1})] = [\mathbf{E}_a(\Delta t/2^k)]^2 + 2[\mathbf{E}_a(\Delta t/2^k)],\ k = m,\ldots,1$$

3) Compute $[\mathbf{G}(\Delta t)]$, $[\dot{\mathbf{G}}(\Delta t)]$, $[\dot{\mathbf{D}}(\Delta t)]$ from

$$[\mathbf{G}(\Delta t)] = [\mathbf{J}_a(\Delta t)] + [\mathbf{I}] + [\mathbf{H}(\Delta t)][\mathbf{M}]^{-1}[\mathbf{C}]$$

$$[\dot{\mathbf{G}}(\Delta t)] = -[\mathbf{H}(\Delta t)][\mathbf{M}]^{-1}[\mathbf{K}]$$

$$[\dot{\mathbf{D}}(\Delta t)] = [\mathbf{H}(\Delta t)]\cdot[\mathbf{M}]^{-1}[\mathbf{f}] + [\mathbf{D}(\Delta t)]\cdot[\mathbf{S}_1]$$

$$\begin{Bmatrix} \mathbf{u}(\Delta t) \\ \mathbf{v}(\Delta t) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}(\Delta t) & \mathbf{H}(\Delta t) \\ \dot{\mathbf{G}}(\Delta t) & \dot{\mathbf{H}}(\Delta t) \end{bmatrix}_{2N_s \times 2N_s} \cdot \begin{Bmatrix} \mathbf{u}_0 - \mathbf{u}_s(0) \\ \mathbf{v}_0 - \mathbf{v}_s(0) \end{Bmatrix} + \begin{Bmatrix} \mathbf{u}_s(\Delta t) \\ \mathbf{v}_s(\Delta t) \end{Bmatrix} \tag{56b}$$

3) The new PTI method by step-response and impulsive-response matrices with expanded dimension

$$\begin{Bmatrix} \mathbf{u}^*(\Delta t) \\ \mathbf{v}^*(\Delta t) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^*(\Delta t) & \mathbf{H}^*(\Delta t) \\ \dot{\mathbf{G}}^*(\Delta t) & \dot{\mathbf{H}}^*(\Delta t) \end{bmatrix}_{2(N_s+g) \times 2(N_s+g)} \cdot \begin{Bmatrix} \mathbf{u}_0^* \\ \mathbf{v}_0^* \end{Bmatrix} \tag{56c}$$

4) The new PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension

$$\begin{Bmatrix} \mathbf{u}(\Delta t) \\ \mathbf{v}(\Delta t) \\ \mathbf{Z}(\Delta t) \end{Bmatrix} = \begin{bmatrix} \mathbf{G}(\Delta t) & \mathbf{H}(\Delta t) & \mathbf{D}(\Delta t) \\ \dot{\mathbf{G}}(\Delta t) & \dot{\mathbf{H}}(\Delta t) & \dot{\mathbf{D}}(\Delta t) \\ 0 & 0 & \mathbf{E}(\Delta t) \end{bmatrix}_{(2N_s+g) \times (2N_s+g)} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \\ \mathbf{Z}_0 \end{Bmatrix} \tag{56d}$$

Assuming that all the matrices are full, the computational efforts to get the results of Eqs. (56a–56d) are studied. For one step, the operation counts of these methods are given in Tables 2–5.

The PTI algorithm by step- and impulsive-response matrices with expanded dimension proposed in this paper will be better than the PTI method with expanded dimension [10] if the following condition is satisfied:

$$2(2N_s + g)^3 + m(2N_s + g)^3 + (2N_s + g)^2 \frac{T}{\Delta t} > 2n \cdot (N_s + g)^3 + 4m \cdot (N_s + g)^3 + 4(N_s + g)^2 \frac{T}{\Delta t} \tag{57}$$

or

$$(N_s + g)^3 + (16 + 4m - 2n) + 6(N_s + g)g^2(2 + m) > g^3(2 + m) + 12(N_s + g)^2 g(2 + m) + (4N_s g + 3g^3)\frac{T}{\Delta t} \tag{58}$$

or

$$(N_s + g) + (16 + 4m - 2n)\frac{1}{(2 + m)} + 6\frac{g^2}{(N_s + g)} > \frac{g^3}{(N_s + g)^2} + 12g + (4N_s g + 3g^3)\frac{T}{\Delta t}\frac{1}{(2 + m)(N_s + g)^2} \tag{59}$$

If $m = 20$ and $n = 4$,

$$4N_s + 6\frac{g^2}{(N_s + g)} > \frac{g^3}{(N_s + g)^2} + 8g + \frac{(2N_s g + 1.5g^3)T}{11\Delta t(N_s + g)^2} \tag{60}$$

Table 2 Computational counts of PTI method in [10]

Category	Operation count
Form $[\mathbf{T}_a^*]$	$2(2N_s + g)^3$
Form $[\mathbf{T}^*]$	$m(2N_s + g)^3$
$[\mathbf{T}^*]\{\mathbf{u}^*\}$	$(2N_s + g)^2$

Table 3 Computational counts of PTI method in [8]

Category	Operation count
Initial Taylor series approximation	$2nN_s^3$
[H] and [J] matrices formulation	$4m \cdot N_s^3$
Particular solution	$(Inte)$
One time-step advancement	$4N_s^2$

Table 4 Operation counts of present PTI method by step-response and impulsive-response matrices with expanded dimension

Category	Operation count
Initial Taylor series approximation	$2n \cdot (N_s + g)^3$
$[\mathbf{H}^*]$ and $[\mathbf{J}^*]$ matrices formulation	$4m \cdot (N_s + g)^3$
One time-step advancement	$4(N_s + g)^2 \frac{T}{\Delta t}$

Table 5 Operation counts of present PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension

Category	Operation count
Initial Taylor series approximation	$2nN_s^3 + nN_s^2 g + ng^3$
[H], [J], [D], and [E] matrices formulation	$4m \cdot N_s^3 + 4m \cdot N_s^2 \cdot g + mg^3$
One time-step advancement	$(4N_s^2 + 2N_s g + g^2)\frac{T}{\Delta t}$

Equation (60) can be simplified as

$$4N_s^3 - N_s\left(6g^2 + \frac{2T}{11\Delta t}g\right) - \left(3 + \frac{3T}{22\Delta t}\right)g^3 > 0 \tag{61}$$

or

$$4N_s^2 > \left(6g^2 + \frac{2T}{11\Delta t}g\right) + \left(3 + \frac{3T}{22\Delta t}\right)\frac{g^3}{N_s} \tag{62}$$

Hence, the PTI algorithm by step and impulsive-response matrices with expanded dimension will be more efficient than the PTI method with expanded dimension [10] if Eq. (62) is satisfied. The value of Δt can be large by using the precise time-step integration method while the accuracy is still maintained. And so, the value of $T/\Delta t$ is normally not very large. Equation (62) can be satisfied easily when the dimension of the system is large. For example, if $g = 6$ and $T/\Delta t = 1.0$, Eq. (62) gives $N_s > 9$.

Furthermore, the PTI algorithm by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension will be more efficient than the PTI method in [8] if

$$2nN_s^3 + nN_s^2 \cdot g + n \cdot g^3 + 4m \cdot N_s^3 + 4m \cdot N_s^2 \cdot g + mg^3 + \left(4N_s^2 + 2N_s \cdot g + 2g^2\right) \cdot \frac{T}{\Delta t} < 2n \cdot N_s^3 + 4m \cdot N_s^3 + 4N_s^2 \frac{T}{\Delta t} + (Inte) \tag{63}$$

or

$$(4m + n)N_s^2 g + (m + n)g^3 + (2gN_s + g^2)\frac{T}{\Delta t} < (Inte) \tag{64}$$

The value of g depends on the external loading form and g in general is not very large. The magnitude of $(Inte)$ is also determined by external loading form. If the excitation is complex, $(Inte)$ will be large. The PTI method with step-response, impulsive-response, and Duhamel-response matrices with expanded dimension will be more efficient than the PTI method in [8] when Eq. (64) is satisfied.

For example, if the external load is linear, $g = 2$, $m = 20$, $n = 4$, and $(Inte)$ would be $N_s^3 + N_s^2 \cdot T/\Delta t$. Equation (64) is then given by

$$(4m + n)N_s^2 g + (m + n)g^3 + (2gN_s + g^2)\frac{T}{\Delta t} < N_s^3 + N_s^2 \cdot \frac{T}{\Delta t} \tag{65}$$

or,

$$N_s^3 - \left(178 - \frac{T}{\Delta t}\right) N_s^2 - 4\frac{T}{\Delta t} N_s - \left(192 + 4\frac{T}{\Delta t}\right) > 0 \quad (66)$$

If $T/\Delta t = 1.0$, then there is $N_s > 177$ from Eq. (66).

From Tables 4 and 5, the present PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension will be more efficient than the PTI method by step- and impulsive-response matrices with expanded dimension if the following equation is satisfied:

$$\begin{aligned} &2nN_s^3 + nN_s^2 \cdot g + n \cdot g^3 + 4m \cdot N_s^3 + 4m \cdot N_s^2 \cdot g + mg^3 \\ &+ \left(4N_s^2 + 2N_s \cdot g + 2g^2\right) \cdot \frac{T}{\Delta t} < 2n \cdot (N_s + g)^3 \\ &+ 4m \cdot (N_s + g)^3 + 4(N_s + g)^2 \frac{T}{\Delta t} \end{aligned} \quad (67)$$

or

$$\begin{aligned} 0 < &\left(5nN_s^2 g + 6nN_s g^2 + ng^3\right) \\ &+ \left(8mN_s^2 g + 12mN_s g^2 + 3mg^3\right) + (6N_s g + 2g^2)\frac{T}{\Delta t} \end{aligned} \quad (68)$$

Note that Eq. (68) is always true. Hence, the computational speed of the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension is always faster than that of the PTI method by step-response and impulsive-response matrices with expanded dimension normally.

In conclusion, the PTI method by step-response and impulsive-response matrices with expanded dimension, and the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension, can reduce much more computational cost than the PTI algorithm with expanded dimension given in [10]. The new PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension proposed here will be more efficient than the PTI method in [8] if Eq. (64) is satisfied. The PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension is more efficient than the other PTI methods discussed in this paper.

VII. Numerical Examples

Two numerical examples are used to illustrate the efficiency of the present algorithms. Computational effort comparisons of the original PTI method [2], the PTI method with expanded dimension [10], the PTI method by step-response and impulsive-response matrices [8], and the two new PTI methods by response matrices with expanded dimension are given in the examples. The validity and the efficiency of the new methods proposed in this paper are demonstrated. All problems are run on a PC with Pentium III 1 GHz CPU.

A. Example 1: Multi-Degree-of-Freedom Systems

Consider a multi-degree-of-freedom system governed by

$$[\mathbf{M}]\{\ddot{\mathbf{u}}\} + [\mathbf{C}]\{\dot{\mathbf{u}}\} + [\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{r}(t)\}, \qquad t \in [0, 10] \quad (69)$$

where $[\mathbf{M}]$, $[\mathbf{C}]$, and $[\mathbf{K}]$ are $N_s \times N_s$ matrices

$$[\mathbf{M}] = \begin{bmatrix} 8 & & & \\ & 8 & & \\ & & \ddots & \\ & & & 8 \end{bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 5 & -5 \\ & & & -5 & 5 \end{bmatrix} \quad (70a)$$

$$[\mathbf{K}] = \begin{bmatrix} 8 & -4 & & & \\ -4 & 8 & -4 & & \\ & \ddots & \ddots & \ddots & \\ & & -4 & 8 & -4 \\ & & & -4 & 4 \end{bmatrix} \quad (70b)$$

where units have been omitted for convenience. The initial displacement and velocity vectors $\{\mathbf{u}_0\}$ and $\{\dot{\mathbf{u}}_0\}$ of length N_s are

$$\begin{aligned} \{\mathbf{u}_0\} &= \{0, \quad 0, \quad \ldots, \quad 0, \quad 1\} \\ \{\dot{\mathbf{u}}_0\} &= \{0, \quad 0, \quad \ldots, \quad 0, \quad 1\} \end{aligned} \quad (71)$$

1. Linear Excitation

Consider a linear external loading form given by

$$\{\mathbf{r}(t)\} = \{\mathbf{r}_0\} + \{\mathbf{r}_1\} \cdot t \quad (72)$$

where $\{\mathbf{r}_0\}$ and $\{\mathbf{r}_1\}$ are $N_s \times 1$ vector composed of the following repeated identical triplets:

$$\{\mathbf{r}_0\} = \{1, \quad 2, \quad 0, \quad 1, \quad 2, \quad 0, \quad \ldots\} \quad (73a)$$

$$\{\mathbf{r}_1\} = \{0.01, \quad 0.02, \quad 0, \quad 0.01, \quad 0.02, \quad 0, \quad \ldots\} \quad (73b)$$

Then,

$$\begin{aligned} &[\mathbf{f}] = [\,\mathbf{r}_0 \quad \mathbf{r}_1\,], \qquad \{\mathbf{Z}(t)\} = \begin{Bmatrix} 1 \\ t \end{Bmatrix} \\ &[\mathbf{S}_1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{S}_2] = [\mathbf{0}] \end{aligned} \quad (74)$$

A time history of the response for a system with $N_s = 200$ is shown in Fig. 1. With time step $\Delta t = 1.0$, all the PTI methods discussed in this paper can give very accurate results. However, the computational efforts are not the same. To illustrate the differences in the computational cost, a larger time step $\Delta t = 10.0$ is considered. The comparison of the computational cost of the original PTI method [4], the PTI method with expanded dimension [10], the PTI method [8], and the two new PTI methods by response matrices with expanded dimension is shown in Table 6. The results show that the computing efficiency can be improved significantly by using the new PTI method proposed in this paper. By making use of the symmetry property, the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension needs much less computational effort.

2. Polynomial Excitation

Consider the following loading form

$$\begin{aligned} \{\mathbf{r}(t)\} = e^{\alpha t}(&\{\mathbf{r}_0\} + \{\mathbf{r}_1\} \cdot t + \{\mathbf{r}_2\} \cdot t^2 + \{\mathbf{r}_3\} \cdot t^3 \\ &+ \{\mathbf{r}_4\} \cdot t^4 + \{\mathbf{r}_5\} \cdot t^5 + \{\mathbf{r}_6\} \cdot t^6) \end{aligned} \quad (75)$$

where $\alpha = 0.01$.

$$\{\mathbf{r}_0\} = \{1, \quad 2, \quad 0, \quad 1, \quad 2, \quad 0, \quad \ldots\} \quad (76a)$$

$$\{\mathbf{r}_1\} = 10^{-1} \times \{1, \quad 2, \quad 0, \quad 1, \quad 2, \quad 0, \quad \ldots\} \quad (76b)$$

$$\{\mathbf{r}_2\} = 10^{-2} \times \{1, \quad 2, \quad 0, \quad 1, \quad 2, \quad 0 \quad \ldots\} \quad (76c)$$

$$\{\mathbf{r}_3\} = 10^{-3} \times \{2, \quad 4, \quad 0, \quad 2, \quad 4, \quad 0, \quad \ldots\} \quad (76d)$$

$$\{\mathbf{r}_4\} = 10^{-4} \times \{2, \quad 4, \quad 0, \quad 2, \quad 4, \quad 0, \quad \ldots\} \quad (76e)$$

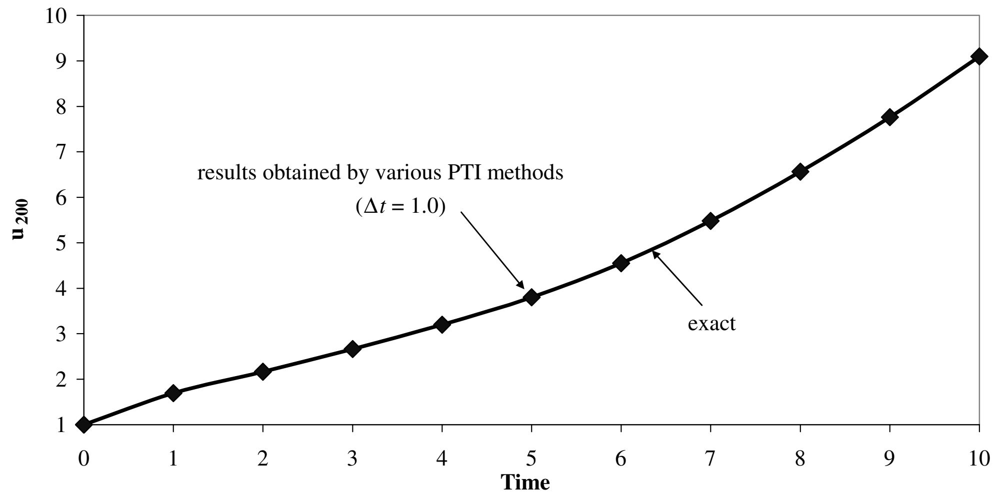

Fig. 1 Time history of u_{200} in Sec. VII.A (with linear excitation).

$$\{\mathbf{r}_5\} = 10^{-5} \times \{2,\quad 4,\quad 0,\quad 2,\quad 4,\quad 0,\quad \ldots\} \tag{76f}$$

$$\{\mathbf{r}_6\} = 10^{-6} \times \{2,\quad 4,\quad 0,\quad 2,\quad 4,\quad 0,\quad \ldots\} \tag{76g}$$

As shown in Sec. III, the matrices $[\mathbf{S}_1]$ and $[\mathbf{S}_2]$ are given by

$$[\mathbf{S}_1] = \begin{bmatrix} \alpha & & & & & & \\ 1 & \alpha & & & & & \\ & 2 & \alpha & & & & \\ & & 3 & \alpha & & & \\ & & & 4 & \alpha & & \\ & & & & 5 & \alpha & \\ & & & & & 6 & \alpha \end{bmatrix}$$
$$[\mathbf{S}_2] = \begin{bmatrix} \alpha^2 & & & & & & \\ 2\alpha & \alpha^2 & & & & & \\ 2 & 4\alpha & \alpha^2 & & & & \\ & 6 & 6\alpha & \alpha^2 & & & \\ & & 12 & 8\alpha & \alpha^2 & & \\ & & & 20 & 10\alpha & \alpha^2 & \\ & & & & 30 & 12\alpha & \alpha^2 \end{bmatrix} \tag{77}$$

A comparison of the computational cost is shown in Table 7. From Table 7, by making use of the symmetry property, the PTI method [8] is faster than the PTI method by step-response and impulsive-response matrices with expanded dimension. However, the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension will cost less than the PTI method [8], because the method proposed in this paper can also make use of the symmetry property of the matrices. Furthermore, the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension will be more efficient than the PTI method [8] with the increase of the number of excitation terms. For example, Table 6 notes that the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension only saves 0.246 s in computational cost compared with the PTI method [8] when the number of excitation terms is two. From Table 7, the saved computational cost increases to 0.918 s when the number of excitation terms is seven.

3. *Sinusoidal Excitation*

Consider the following loading form:

$$\{\mathbf{r}(t)\} = \sum_{i=1}^{3}\{\mathbf{r}_i\}t^{i-1}(\sin(t)+\cos(t)) = \{\mathbf{r}_1\}\sin(t) + \{\mathbf{r}_1\}\cos(t) + \{\mathbf{r}_2\}t\sin(t) + \{\mathbf{r}_2\}t\cos(t) + \{\mathbf{r}_3\}t^2\sin(t) + \{\mathbf{r}_3\}t^2\cos(t) \tag{78}$$

where

$$\{\mathbf{r}_1\} = \{1,\quad 2,\quad 0,\quad 1,\quad 2,\quad 0,\quad \ldots\} \tag{79a}$$

$$\{\mathbf{r}_2\} = \{0.1,\quad 0.2,\quad 0,\quad 0.1,\quad 0.2,\quad 0,\quad \ldots\} \tag{79b}$$

$$\{\mathbf{r}_3\} = \{0.01,\quad 0.02,\quad 0,\quad 0.01,\quad 0.02,\quad 0,\quad \ldots\} \tag{79c}$$

As shown in Sec. III, the matrices $[\mathbf{S}_1]$ and $[\mathbf{S}_2]$ are given by

Table 6 Computational effort to evaluate result u_{N_s} at $t = 10$ s with $\Delta t = 10.0$ s and linear excitation in Sec. VII.A ($n = 4$, $m = 20$, $\Delta t = 10.0$ s, $N_s = 200$)

Methods	Order of computing equation	Result u_{N_s}	Computational cost, s
Original PTI method [4]	First-order	9.09462099	149.475
PTI method with expanded dimension [10]	First-order	9.09462099	111.520
PTI method by step-response and impulsive-response matrices [8]	Second-order	9.09462099	53.492
PTI method by step- and impulsive-response matrices with expanded dimension	Second-order	9.09462099	90.319
PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension	Second-order	9.09462099	53.246

Table 7 Computational effort to evaluate result u_{N_s} at $t = 10$ s with $\Delta t = 10.0$ s and polynomial excitation in Sec. VII.A ($n = 4, m = 20, \Delta t = 10.0$ s, $N_s = 200$)

Methods	Order of computing equation	Result u_{N_s}	Computational cost, s
PTI method with expanded dimension [10]	First-order	18.18160768	114.665
PTI method by step-response and impulsive-response matrices [8]	Second-order	18.18160768	54.304
PTI method by step-response and impulsive-response matrices with expanded dimension	Second-order	18.18160768	97.730
PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension	Second-order	18.18160768	53.386

Table 8 Computational effort to evaluate result u_{N_s} at $t = 10$ s with $\Delta t = 10.0$ s and complex excitation in Sec. VII.A ($n = 4, m = 20, \Delta t = 10.0$ s, $N_s = 200$)

Methods	Order of computing equation	Result u_{N_s}	Computational cost, s
PTI method with expanded dimension [10]	First-order	3.49271662	112.060
PTI method by step-response and impulsive-response matrices with expanded dimension	Second-order	3.49271662	95.826
PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension	Second-order	3.49271662	53.306

$$[\mathbf{S}_1] = \begin{bmatrix} 0 & 1 & & & & 0 \\ -1 & 0 & 0 & & & \\ 1 & 0 & 0 & 1 & & \\ & 1 & -1 & 0 & 0 & \\ & & 2 & 0 & 0 & 1 \\ 0 & & & 2 & -1 & 0 \end{bmatrix}$$
$$[\mathbf{S}_2] = \begin{bmatrix} -1 & 0 & & & & 0 \\ 0 & -1 & 0 & & & \\ 0 & 2 & -1 & 0 & & \\ -2 & 0 & 0 & -1 & 0 & \\ 2 & 0 & 0 & 4 & -1 & 0 \\ 0 & 2 & -4 & 0 & 0 & -1 \end{bmatrix} \quad (80)$$

For such a complex external loading form, the particular solution will be complicated. This example also compares the efficiency of the current two new methods. The results are shown in Table 8. The symmetry property can still be used in the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension. The PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension is more efficient than the PTI method by step-response and impulsive-response matrices with expanded dimension.

B. Example 2: Truss Structure with Six Layers and Three Lattices

Figure 2 shows a truss with six layers and three lattices. It has 28 nodal points, 60 truss elements, and 48 degrees of freedom. The Young's modulus of each element, whose transverse surface area is 5×10^{-3} m^2, is 200 GPa. In the truss structure, the height of each layer and the width of each lattice are 3 m and 4 m, respectively. The lumped mass at each node of the truss is $M_i = 500$ kg ($i = 5, 6, \ldots, 28$). The loading $r(t)$ is a trapezoidal function as shown in the Fig. 2, i.e.,

$$R(t) = \begin{cases} \frac{6h}{T}(t - iT) & iT - \frac{T}{6} < t < iT + \frac{T}{6} \\ h & iT + \frac{T}{6} \le t \le iT + \frac{T}{3} \\ -\frac{6h}{T}\left(t - \frac{2i+1}{2}T\right) & \frac{2i+1}{2}T - \frac{T}{6} < t < \frac{2i+1}{2}T + \frac{T}{6} \\ -h & \frac{2i+1}{2}T + \frac{T}{6} \le t \le \frac{2i+1}{2}T + \frac{T}{3} \end{cases},$$
$$i = 0, 1, 2, \ldots \quad (81)$$

The excitation can be approximated by a Fourier series [10] as

$$r(t) \approx \sum_{i=1}^{p} a_i[\sin(\omega_i t)] \quad (82)$$

where

$$a_i = \frac{12h}{(2i-1)^2}\sin\left(\frac{2i-1}{3}\pi\right)$$

and

$$\omega_i = (2i-1)\frac{2\pi}{T}$$

In this example, $h = 10^4$, $T = 6.0$ s.

Normally, if the periodic excitation is described by piecewise functions, the current PTI methods would approximate the external load by Fourier series. The accuracy and efficiency therefore depends on the number of Fourier terms being used.

The transient responses under periodic excitations can be evaluated efficiently if the computing matrices need not be computed for every period. In the present formulation, the excitation can be separated into two parts $[\mathbf{F}]$ and $\{\mathbf{Z}(t)\}$ in every segment of the piecewise excitation $\{\mathbf{r}(t)\}$. The matrix $[\mathbf{S}_1]$ (or $[\mathbf{S}_2]$) is determined by the time-varying function $\{\mathbf{Z}(t)\}$. If the function $\{\mathbf{Z}(t)\}$ could be modified so that the matrices $[\mathbf{F}]$ and $[\mathbf{S}_1]$ (or $[\mathbf{S}_2]$) remain unchanged in different periods, the response matrices need not be reevaluated for every period. Hence, the response matrices evaluated in the first period can be stored for other periods. Only initial conditions of $\{\mathbf{Z}(t)\}$ may need to be modified.

In the following, two periodic external loading forms (continuous and discontinuous) are considered. In Fig. 2, the general formula for the first and fifth part of the piecewise function in every period is

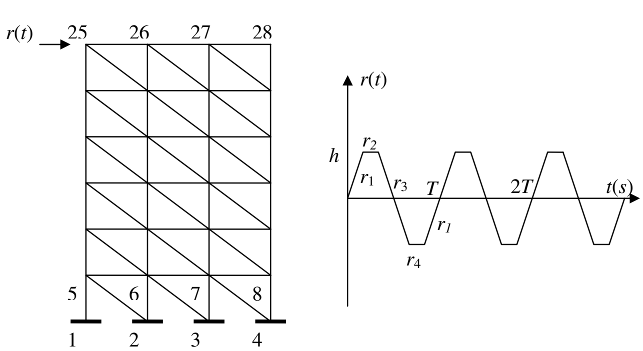

Fig. 2 Truss with six layers and three lattices, and the excitation.

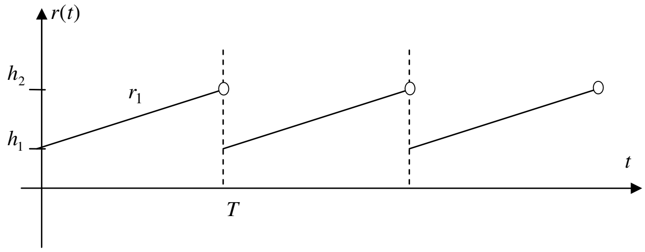

Fig. 3 Discontinuous periodic external loading form.

$$r_1 = \frac{6h}{T}(t - iT)$$

$$iT - \frac{T}{6} < t < iT + \frac{T}{6}, \quad \text{where } T \text{ is the period and } i = 0, 1, 2, \ldots \tag{83}$$

Let $t^* = t - iT$, then r_1 can be rewritten as $r_1 = (6h/T) \cdot t^*$. The matrix $[\mathbf{F}]$ and $[\mathbf{S}_1]$ (or $[\mathbf{S}_2]$) are the same in every period when the variable t^* is used instead of variable t. Hence, the computing matrices only need to be computed four times at most.

For discontinuous periodic external load, this transformation is more useful because it is more difficult to get a good approximation by using just a few Fourier terms. Figure 3 shows a simple discontinuous excitation. The general formula can be written as

$$r_1 = h_1 + \frac{h_2 - h_1}{T} t^*, \quad \text{where } t^* = t - iT \quad \text{and} \quad i = 0, 1, 2, \ldots \tag{84}$$

The computing matrices calculated in the first period can be used in the remaining computational process. The only disadvantage is that the time-step size must be restricted according to the period. This method can work well when the number of the subfunctions in the piecewise function in one period is not too many. In fact, this method can be used to reduce the computational cost in all PTI methods with expanded dimension.

The time-step integration procedures used are the same as those in Sec. VII.A. For periodic excitation, more computational cost can be saved by using the periodic property instead of Fourier series approximation to the excitation. Here, only four sets of response matrices need be evaluated in the whole computing process. The computational costs are shown in Table 9. Although the two PTI methods only do one step calculation with Fourier series approximation, the computational effort still is very high. In the PTI method with expanded dimension and the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension, the computational costs by using Fourier series approximation are 17.815 and 3.995 s, and are about five and six times more than the cost by using the periodic property, respectively. This example also further demonstrates that the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension are more efficient than the PTI method with expanded dimension [10].

VIII. Conclusions

The PTI method in [8] is further developed in this paper. The second-order equations are tackled directly without transforming to a system of equivalent first-order equations first. To simplify the computational algorithm, the nonhomogeneous second-order equations are transformed into homogeneous second-order equations by expanding the dimension of the matrices. The PTI method by step-response and impulsive-response matrices, and the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension, are proposed in this paper. These two new PTI methods are more efficient than the original PTI method in [8] and the PTI method with expanded dimension in [10]. In the PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension, the symmetry property can still be used to reduce much computational cost. Numerical examples have been used to illustrate the improvement in computational efficiency.

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Table 9 Computational effort to evaluate results at the end of two periods ($t = 12.0$ s) in Sec. VII.B ($n = 4, m = 20$, relative error 10^{-7}, p is number of terms of Fourier series)

Excitation form	Time-step size	Computational cost, s: PTI method with expanded dimension [10]	Computational cost, s: PTI method by step-response, impulsive-response, and Duhamel-response matrices with expanded dimension
Fourier series approximation [10] with $p = 60$	$\Delta t = 12.0$ s	17.815	3.995
With periodic property in Sec. VII.B	$\Delta t = 1.0$ s	3.595	0.671

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